Higher-gradient theories for fluids and concentrated effects

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Outline of the presentation

Higher-order theories: *why not?*

- A new equation of motion
- Constitutive theory
- Well-posed differential problems

Higher-order theories: *why?*

- General set up for dynamical systems
- Higher-order power expenditures
- Constitutive role of kinematics

Second-gradient liquids

FRIED & GURTIN, *Arch. Ration. Mech. Anal.*, 2006

$$\int_\Omega \mathbf{T_u} \cdot \nabla \mathbf{v} + \int_\Omega \mathbf{G_u} \cdot \nabla\nabla \mathbf{v} = \int_\Omega \rho(\mathbf{b} - \dot{\mathbf{u}}) \cdot \mathbf{v}$$

- Mechanical constraint: $\operatorname{div} \mathbf{u} = \operatorname{div} \mathbf{v} = 0$
- A new length-scale appears: $[\mathbf{G_u}] = [\mathit{length}][\mathbf{T_u}]$
- A legitimate choice

Linear, isotropic, incompressible, homogeneous

MUSESTI, *Acta Mech.*, 2009

$$\mathbf{T_u} = 2\mu \,\mathrm{Sym}\, \nabla\mathbf{u} - p\,\mathbf{I}$$

$$\mathbf{G_u} = (\eta_1 - \eta_2)\nabla\nabla\mathbf{u} + 3\eta_2 \,\mathrm{Sym}\, \nabla\nabla\mathbf{u}$$
$$- (\eta_2 + 5\eta_3)\Delta\mathbf{u} \otimes \mathbf{I} + 3\eta_3 \,\mathrm{Sym}\,(\Delta\mathbf{u} \otimes \mathbf{I}) - \mathbf{I} \otimes \mathbf{p}$$

Three new material parameters such that

$$\frac{[\eta_i]}{[\mu]} = [length]^2$$

Thermodynamically consistent

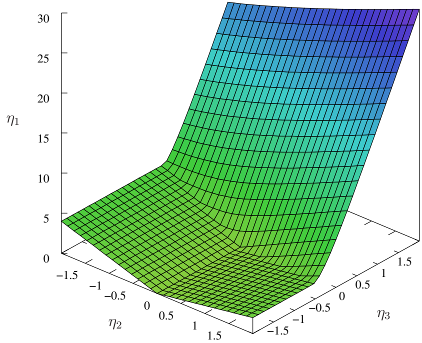
The instant dissipation must be nonnegative for any flow

$$\Updownarrow$$

$$\mu \geq 0$$

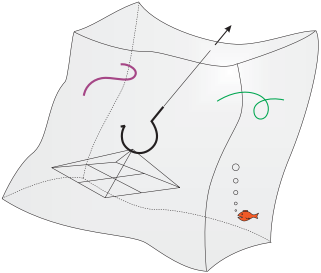
$$\eta_1 + 2\eta_2 \geq 0$$

$$\eta_1 - \eta_2 \geq 0$$



$$\eta_1 - \eta_2 - 6\eta_3 - 2\sqrt{\eta_2^2 + 4\eta_2\eta_3 + 9\eta_3^2} \geq 0$$

1D rigid bodies in a 3D region



Situations:

1. A pressure gradient drives the flow producing in- and out-flows through some parts of the boundary
2. A closed container where thin objects swim

The thin moving objects are rigid bodies with finite 1D Hausdorff measure, so that their motion plainly satisfies the incompressibility condition, though they cannot swim.

Existence and uniqueness of flows

Giusteri, Marzocchi, Musesti
Mech. Res. Commun., 2010 – *Acta Mech.*, 2011

- H and H^2_d: completion in L^2 and in H^2 of divergence-free smooth vector fields
- $\Lambda(t)$: position of the 1D rigid body at time t
- $X_t := \{ \mathbf{v} \in H^2(\Omega; \mathbb{R}^3) : \mathbf{v} = 0 \text{ on } \partial\Omega \cup \Lambda(t) \} \cap H^2_d$
- divergence-free interpolator $\mathbf{w}$ for the velocity on $\Lambda(t)$

For every initial datum $\mathbf{u}_0 \in H$, there exists a unique

$$\mathbf{u} \in L^2([0,T]; X_t) \cap C^0([0,T]; H) \cap H^1([0,T]; X_t') =: \mathcal{X} ,$$

such that the field $\mathbf{u} + \mathbf{w}$ represents the considered flow.

Analytical technique

- A suitable function space to model adherence
- Mapping to a time-independent space, with a time-dependent differential operator
- Compact nonlinearity:

$$\dot{\mathbf{u}} = \frac{\partial \mathbf{u}}{\partial t} + (\nabla \mathbf{u})\mathbf{u}$$

- Lax-Milgram provided $\mu > 0$ and $\eta_1 - \eta_2 > 0$
- Fixed-point theorem based on *a priori* estimates
- Uniqueness thanks to the essential boundedness of functions in $\mathcal{X}$

Geometrical approach to dynamical systems

A dynamical system is a triple $(\Omega, \mathcal{D}, \mathcal{S})$ where:

1. the set Ω is called *underlying space*;
2. the *phase space* $\mathcal{D}$ is a product of sets of functions, elements of $\mathcal{D}$ are called *configurations*, and their components are the *descriptors* of the system;
3. $\mathcal{D}$ is a Banach Manifold;
4. $\mathcal{S}$ is a section of the cotangent bundle on $\mathcal{D}$.

A point $u \in \mathcal{D}$ is an *equilibrium configuration* if

$$\langle \mathcal{S}_u, v \rangle = 0$$

for every *generalized virtual velocity* $v \in T_u\mathcal{D}$.

This condition is called *Principle of Virtual Powers*.

Generality and flexibility

- Both finite and infinite dimensional, depending on $\mathcal{D}$
- Both steady and evolutionary problems

$$\langle \frac{\partial u}{\partial t}, v(t) \rangle = \langle \mathcal{S}_{u(t)}, v(t) \rangle \quad \longleftrightarrow \quad \langle \tilde{\mathcal{S}}_{\tilde{u}}, \tilde{v} \rangle = 0$$

- Kinematical prescriptions on $\mathcal{D}$ are fundamental: if $\mathcal{D}$ is a Banach space, $T_u\mathcal{D} \cong \mathcal{D}$
- Constitutive prescriptions involve $\mathcal{S}$, but its representation is also dictated by kinematics
- Emphasis is shifted from Ω to $\mathcal{D}$

Non-smooth domains

1. In a Lagrangian perspective, topological properties of Ω are relevant, and the *placement* is a fundamental descriptor;
2. in an Eulerian perspective, properties of Ω are less important: phase spaces with a similar structure can be defined on domains with very different structures.

The interrelation between the Lebesgue measure and the Euclidean distance, on which the classical Sobolev spaces are built, can be reproduced on poorer *metric measure spaces*.

This allows for the generalization to non-smooth domains of continuum mechanical theories based on Sobolev spaces.

A definition for higher-order power expenditures

Take $\Omega \subseteq \mathbb{R}^n$ open, and assume that

$$\mathcal{D} = H^{k_1}(\Omega) \times \ldots \times H^{k_m}(\Omega)\,.$$

A $(k_1, \ldots, k_m)$-*power* is a section $\mathcal{P}$ of the cotangent bundle $T^*\mathcal{D}$ such that, for any $v = (v_s)_{s=1}^m \in T_u\mathcal{D}$,

$$\langle \mathcal{P}_u, v \rangle = \sum_{s=1}^{m} \sum_{i=0}^{k_s} \int_\Omega A_u^{(i,s)} \cdot \nabla^i v_s\,,$$

for some vector fields $A^{(i,s)}$ on $\mathcal{D}$, with values in $L^2(\Omega; \mathbb{R}^{n^i})$.

Representation on piecewise smooth domains

Let $\Omega \subseteq \mathbb{R}^3$ be an open bounded domain with piecewise smooth boundary, and let $\mathcal{E}$ denote the singular part of $\partial\Omega$; then there exist tensor fields $\hat{\mathbf{b}}_{\mathbf{u}}$, $\hat{\mathbf{t}}_{\mathbf{u}}$, and, if $k \geq 2$, $\{\hat{\mathbf{M}}_{\mathbf{u}}^{(s)}\}_{s=0}^{k-2}$ and $\{\hat{\mathbf{K}}_{\mathbf{u}}^{(s)}\}_{s=0}^{k-2}$, such that

$$\sum_{i=0}^{k} \int_\Omega \mathbf{A}_{\mathbf{u}}^{(i)} \cdot \nabla^i \mathbf{v} = \int_\Omega \hat{\mathbf{b}}_{\mathbf{u}} \cdot \mathbf{v} + \int_{\partial\Omega} \hat{\mathbf{t}}_{\mathbf{u}} \cdot \mathbf{v} + \sum_{s=0}^{k-2} \int_{\partial\Omega} \hat{\mathbf{M}}_{\mathbf{u}}^{(s)} \cdot \frac{\partial}{\partial n}(\nabla^s \mathbf{v}) + \sum_{s=0}^{k-2} \int_{\mathcal{E}} \hat{\mathbf{K}}_{\mathbf{u}}^{(s)} \cdot \nabla^s \mathbf{v}\,.$$

Interaction fields

$$\hat{\mathbf{b}}_{\mathbf{u}} = \sum_{i=0}^{k} (-\operatorname{div})^{i} \mathbf{A}_{\mathbf{u}}^{(i)}$$

$$\hat{\mathbf{t}}_{\mathbf{u}} = \sum_{i=1}^{k} \sum_{j=1}^{i} (-\operatorname{div}_{\mathcal{S}})^{(j-1)} \{ [(-\operatorname{div})^{(i-j)} \mathbf{A}_{\mathbf{u}}^{(i)}] \mathbf{n} \}$$

$$\hat{\mathbf{M}}_{\mathbf{u}}^{(s)} = \sum_{i=s}^{k-2} \sum_{j=0}^{i-s} \{ (-\operatorname{div}_{\mathcal{S}})^{(j)} [[(-\operatorname{div})^{(i-j-s)} \mathbf{A}_{\mathbf{u}}^{(i+2)}] \mathbf{n}] \} \mathbf{n}$$

$$\hat{\mathbf{K}}_{\mathbf{u}}^{(s)} = \sum_{i=s}^{k-2} \sum_{j=0}^{i-s} \{ (-\operatorname{div}_{\mathcal{S}})^{(j)} [[(-\operatorname{div})^{(i-j-s)} \mathbf{A}_{\mathbf{u}}^{(i+2)}] \mathbf{n}_{a}] \} \mathbf{e}_{a} + \sum_{i=s}^{k-2} \sum_{j=0}^{i-s} \{ (-\operatorname{div}_{\mathcal{S}})^{(j)} [[(-\operatorname{div})^{(i-j-s)} \mathbf{A}_{\mathbf{u}}^{(i+2)}] \mathbf{n}_{b}] \} \mathbf{e}_{b}$$

The second-order case

Degiovanni, Marzocchi, Musesti
Ann. Mat. Pura Appl., 2006

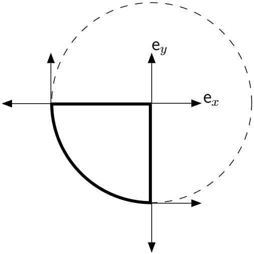
$$\hat{\mathbf{b}}_{\mathbf{u}} = \mathbf{A}_{\mathbf{u}}^{(0)} - \operatorname{div}\mathbf{A}_{\mathbf{u}}^{(1)} + \operatorname{div}\operatorname{div}\mathbf{A}_{\mathbf{u}}^{(2)}$$

$$\hat{\mathbf{t}}_{\mathbf{u}} = [\mathbf{A}_{\mathbf{u}}^{(1)} - \operatorname{div}\mathbf{A}_{\mathbf{u}}^{(2)}]\mathbf{n} + \operatorname{div}_{\mathcal{S}}[\mathbf{A}_{\mathbf{u}}^{(2)}\mathbf{n}]$$

$$\hat{\mathbf{m}}_{\mathbf{u}} = \mathbf{A}_{\mathbf{u}}^{(2)}[\mathbf{n}\otimes\mathbf{n}]$$

$$\hat{\mathbf{k}}_{\mathbf{u}} = \mathbf{A}_{\mathbf{u}}^{(2)}[\mathbf{e}_a\otimes\mathbf{n}_a + \mathbf{e}_b\otimes\mathbf{n}_b]$$

Concentrated or diffused interactions



- Ansatz: $\mathbf{u} = u(r)\mathbf{e}_z$
- $L := \sqrt{(\eta_1 - \eta_2 - 4\eta_3)/\mu}$
- Solution:

$$u(r) = \alpha_1 + \alpha_2 I_0(r/L) + \alpha_3 \left[\log(r/L) + K_0(r/L)\right]$$

- B.C. and adherence: $u(0) = U\,,\ u(R) = 0\,,$

$$\left[\eta_1 \frac{\partial^2 u}{\partial r^2} - (\eta_2 + 4\eta_3)\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)\right]_R = 0$$

- Concentrated stress: $\hat{\mathbf{k}}_{\mathbf{u}} \propto \eta_1 \dfrac{\partial^2 u}{\partial x \partial y}$, independent of η_3

Constitutive lack of concentrated stresses

PODIO-GUIDUGLI & VIANELLO
Contin. Mech. Thermodyn., 2010

If the second-order tensor takes the form

$$\mathbf{G_u} = \beta(\mathbf{g_u} \otimes \mathbf{I}) + \textit{pressure terms}$$

for some vector field $\mathbf{g_u}$, then $\hat{\mathbf{k}}_\mathbf{u} = 0$, and there is no concentrated interaction along edges.

This is the case when $\eta_1 = \eta_2 = 0$, $\eta_3 < 0$, with $\mathbf{g_u} = \Delta\mathbf{u}$.

A further differential problem

With $\eta_1 = 0$ coercivity on H^2 is lost, and also concentrated stresses disappear.
Can the adherence to 1D bodies still be modeled?

1. Choose a suitable phase space

$$\mathcal{D} := L^2([0,T];Y) \cap C^0([0,T];Y) \cap H^1([0,T];Y')$$

$$Y := \left\{ \mathbf{u} \in H^1_d : \Delta\mathbf{u} \in L^2 \text{ and } \mathbf{u} = 0 \text{ on } \partial\Omega \cup \Lambda(t) \right\}$$

2. Constitutive prescriptions with $\eta_1 = 0$ and $\eta_3 < 0$ define an integral representation of a section of $T^*\mathcal{D}$
3. Estimates can be proved, providing existence of flows
4. Continuity of functions in Y permits to model adherence, and essential boundedness gives uniqueness

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